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Pole Assignment for Optimal Regulators with a Nonnegative Definite Weighting Matrix

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I. Introduction

POLE placement for the optimal regulator or linear quadratic regulator (LQR) is an approach to linear control system design, which provides both large gain and phase margins by the LQR and desired transients by pole placement. Its goal is to find a weighting matrix or feedback-gain matrix for the LQR that gives prescribed closed-loop eigenvalues (CLEs). Many papers[1–4] on this problem, including the references in Ref. 5, have been published.

We have already proposed a method.[5] Its feature is the integration schemes to combine the closed-loop pole profiles with respect to the elemental variations in the weighting matrix. Because the weighting matrix is diagonal, it is simple to understand physical meaning of the weighting elements besides the behavior of the closed-loop system. Although the method establishes the CLE profiles for a range of diagonal elements in Q, the optimality conditions will prevail for a positive range of diagonal elements. In other words, the method will not guarantee the best attributes of an optimal controller, unless the weighting matrix is preserved as nonnegative definite. We modify the method so that the nonnegative definiteness can be retained by controlling eigenvalues of a nondiagonal weighting matrix. To illustrate its effectiveness, the method is applied to an F-4 aircraft model.

II. Pole Assignment by Continuous Pole Shifting

LQR Problem

For a linear system described by

$$\dot{x} = Ax + Bu \tag{1}$$

a control law is determined so that the performance index

$$J = \int_0^\infty (\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u})\, \mathrm{d}t \tag{2}$$

can be minimized, where $\boldsymbol{x} \in R^n$ is a state vector and $\boldsymbol{u} \in R^r$ a control vector. Here, $\boldsymbol{A} \in R^{n\times n}$ and $\boldsymbol{B} \in R^{n\times r}$ are constant matrices, $\boldsymbol{Q} \in R^{n\times n}$ is a nonnegative definite matrix, and $\boldsymbol{R} \in R^{r\times r}$ is a positive definite one. The pair $(\boldsymbol{A}, \boldsymbol{B})$ is assumed to be a controllable pair.

Characteristic Equation of Hamiltonian Matrix

The following matrix, $\boldsymbol{H} \in R^{2n\times 2n}$, is called a Hamiltonian matrix:

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T \\ -\boldsymbol{Q} & -\boldsymbol{A}^T \end{bmatrix} \tag{3}$$

The characteristic equation of $\boldsymbol{H}$ is

$$\det(\lambda \boldsymbol{I}_{2n} - \boldsymbol{H}) = 0 \tag{4}$$

where $\det(\cdot)$ stands for a determinant, $\boldsymbol{I}_{2n}$ is a $2n \times 2n$ identity matrix, and λ is an eigenvalue of $\boldsymbol{H}$. As is well known, the stable eigenvalues of $\boldsymbol{H}$ are CLEs of the LQR.

Derivation of Differential Equations

The method proposed in Ref. 5 is modified for the nondiagonal weighting matrix as follows. Let us define $f(\boldsymbol{Q}, \lambda_i)$ as the left-hand side of Eq. (4) for a CLE $\lambda_i (i = 1, 2, \ldots, n)$. Let $\boldsymbol{Q} = \{q_{ij}\}$ $(i, j = 1, \ldots, n)$ and $\boldsymbol{R}$ be nominal weighting matrices, and the resulting CLEs for them are $\Lambda = [\lambda_1, \ldots, \lambda_n]^T$. Consider small perturbations, $\Delta\boldsymbol{Q} = \{\Delta q_{ij}\}$ and $\Delta\Lambda = [\Delta\lambda_1, \ldots, \Delta\lambda_n]^T$, from the nominal values $\boldsymbol{Q}$ and Λ, respectively. Define $\Delta\lambda_i$ as $\Delta\lambda_i = (\lambda_i^* - \lambda_{0i})/N$, where λ_i^* and λ_{0i} are desired and initial eigenvalues, respectively, and N is a large integer. The initial eigenvalues are given by an appropriate initial weighting matrix $\boldsymbol{Q}_0$. Further define $\Delta\lambda = \min_i |\Delta\lambda_i|$ and $k_i = \Delta\lambda_i/\Delta\lambda$. Then by applying Taylor series expansion to $f(\boldsymbol{Q}, \lambda_i)(i = 1, \ldots, n)$ around the nominal $\boldsymbol{Q}$ and λ_i and taking Eq. (4) into account, the following equations are obtained using first-order approximation:

$$\frac{\partial f}{\partial \boldsymbol{q}^T}\Delta \boldsymbol{q} = -\boldsymbol{g}\Delta\lambda \qquad \text{or} \qquad \frac{\Delta \boldsymbol{q}}{\Delta\lambda} = -\left[\frac{\partial f}{\partial \boldsymbol{q}^T}\right]^+ \boldsymbol{g} \tag{5}$$

where the superscript + indicates pseudoinverse,

$$f = [f_1, \ldots, f_n]^T$$

$$\boldsymbol{q} = [q_{11}, \ldots, q_{jk}, \ldots, q_{nn}]^T \qquad (j = 1, \ldots, n;\ j \le k \le n)$$

$$\boldsymbol{g} = \left[\frac{\partial f_1}{\partial\lambda_1}k_1, \ldots, \frac{\partial f_n}{\partial\lambda_n}k_n\right]^T, \qquad f_i = f(\boldsymbol{Q}, \lambda_i)$$

Integrating Eq. (5) from λ_{0i} to $\lambda_i^* (i = 1, \ldots, n)$, we get a weighting matrix for the desired CLEs. Throughout the integration, the weighting matrix $\boldsymbol{R}$ for control effort is kept as a constant matrix.

Remark 1 [singularity by open-loop eigenvalues (OLEs)]. In Ref. 5, instead of Eq. (4), a reduced type of the characteristic equation of $\boldsymbol{H}$ is used, whereas in this paper the regular one, Eq. (4), is used. Although both methods give the same result, each has merits and demerits. In the former method, if the CLEs include the OLEs, the derivative matrix $\partial f/\partial \boldsymbol{q}^T$ in Eq. (5) cannot be of full rank, so that singularity arises in its pseudoinverse. By contrast, the latter method does not cause the singularity problem. However, the first method is more computationally efficient than the second one, because the dimension of the matrix in the characteristic equation is n; on the other hand, it is $2n$ in the second method. Hence, it would be better to use the first method, if the CLEs do not include the OLEs throughout the integration.

Remark 2 (uniqueness of the solution). In Ref. 5, because $\partial f/\partial \boldsymbol{q}^T$ is a square matrix with respect to the diagonal weighting matrix, the procedure provides a unique solution. However, in the proposed method, because we adjust all of the elements of the weighting matrix, $\partial f/\partial \boldsymbol{q}^T$ is not a square matrix. The solution is observed to be dependent on the initial weighting matrix $\boldsymbol{Q}_0$. This is an observation made during computer simulations.

III. Control of Eigenvalues of Weighting Matrix

A problem with the above method is that it does not always give a nonnegative definite weighting matrix. In this section, the method is modified to obtain a nonnegative definite weighting matrix by controlling its eigenvalues.

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Differential Equations of Weighting Matrix

The characteristic equation of the weighting matrix $\boldsymbol{Q}$ is given by

$$\det(\lambda_{Qi}\boldsymbol{I}_n - \boldsymbol{Q}) = 0 \tag{6}$$

where λ_{Qi} is an eigenvalue of $\boldsymbol{Q}$. Equation (6) can be differentiated with respect to λ_{Qi} in the same way as Eq. (5). Let the left-hand side of Eq. (6) be $h_i(\boldsymbol{Q}, \lambda_{Qi})$, and the differential equation can be written as

$$\frac{\partial h_i}{\partial \boldsymbol{q}^T}\Delta \boldsymbol{q} = -g_{Qi}\Delta\lambda \tag{7}$$

where g_{Qi} is defined as $g_{Qi} = (\partial h_i/\partial\lambda_{Qi})k_{Qi}$; $k_{Qi} = \Delta\lambda_{Qi}/\Delta\lambda$; $\Delta\lambda_{Qi} = (\lambda^*_{Qi} - \lambda_{Q0i})/N$; λ^*_{Qi} and λ_{Q0i} are desired and initial values of λ_{Qi}, respectively. Suppose that at a point on the integral contour the eigenvalue λ_{Qi} takes a small positive value and a negative gradient, i.e., $\Delta\lambda_{Qi}/\Delta\lambda = (\partial h_i/\partial \boldsymbol{q}^T)\Delta\boldsymbol{q}/(\partial h_i/\partial\lambda_{Qi}) < 0$. Then set $\Delta\lambda_{Qi}$ or g_{Qi} in Eq. (7) to zero and combine it with Eq. (5) as

$$\frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{q}^T}\Delta\boldsymbol{q} = -\boldsymbol{G}\Delta\lambda \quad \text{or} \quad \frac{\Delta\boldsymbol{q}}{\Delta\lambda} = -\left[\frac{\partial\boldsymbol{\eta}}{\partial\boldsymbol{q}^T}\right]^+ \boldsymbol{G} \tag{8}$$

where $\boldsymbol{\eta} = [\boldsymbol{f}^T, h_i]^T$ and $\boldsymbol{G} = [\boldsymbol{g}^T, 0]^T$. Every time an eigenvalue is becoming zero, another differential equation similar to Eq. (7) is combined with Eq. (8), setting the right-hand side of Eq. (7) to zero; as a result, h_i in $\boldsymbol{\eta}$ and 0 in $\boldsymbol{G}$ become vectors. However, the method does not allow λ_{Qi} as well as the CLEs to be repeated eigenvalues, because the derivatives of both sides of the corresponding row of Eq. (8) are zero for the repeated eigenvalues. For example, if λ_i is a repeated eigenvalue, then $\partial\eta_i/\partial\boldsymbol{q}^T = 0$ and $\partial\eta_i/\partial\lambda_i = 0$. Consequently, the rank deficiency of the matrix $\partial\boldsymbol{\eta}/\partial\boldsymbol{q}^T$ causes singularity to its pseudoinverse. In fact, when λ_{Qi} is already kept a small positive value and another eigenvalue is becoming zero, the integration does not proceed. To avoid this problem, we introduce thresholds that are distinct nonnegative values, such as $[0, \varepsilon, 2\varepsilon, 3\varepsilon, \ldots]^T$, where ε is a small positive value. Thereby the eigenvalues are made constant slightly before crossing the thresholds.

Remark 3 (normalization). The magnitude of the elements of the derivative $\partial\boldsymbol{h}/\partial\boldsymbol{q}^T$ is sometimes very small, compared with those of $\partial\boldsymbol{f}/\partial\boldsymbol{q}^T$. This makes the derivative matrix $\partial\boldsymbol{\eta}/\partial\boldsymbol{q}^T$ ill-conditioned. Subsequently, the pseudoinverse in Eq. (8) will be very sensitive to numerical computations. Therefore, it is recommended that each row be normalized by its largest absolute element.

Remark 4 (existence of the solution). The method proposed does not always guarantee the existence of a nonnegative definite weighting matrix. It is presumed that this method fails because of an improper set of desired CLEs. Generally, pole-assignable region by nonnegative definite weighting matrices is restricted.[3] In our approach, this aspect is observed while numerical integration is being performed. That is, as the number of eigenvalues that are constrained at constant values increases, the magnitude of the derivatives tends to become large; as a result, numerical integration becomes inaccurate or does not proceed to retain accuracy.

IV. Numerical Example

The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of a lateral-directional linearized model of the F-4 aircraft[4] are given, respectively, by

$$\boldsymbol{A} = \begin{bmatrix} -0.764 & 0.387 & -12.9 & 0 & 0.952 & 6.05 \\ 0.024 & -0.174 & 4.31 & 0 & -1.76 & -0.416 \\ 0.006 & -0.999 & -0.0578 & 0.0369 & 0.0092 & -0.0012 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

and

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix}^T$$

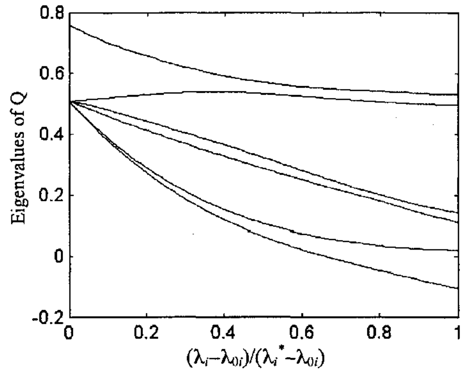

Fig. 1 Trajectories of eigenvalues of the weighting matrix (without adjusting eigenvalues of $\boldsymbol{Q}$).

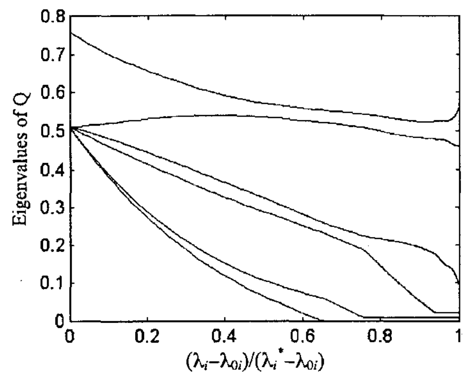

Fig. 2 Trajectories of eigenvalues of the weighting matrix (adjusting eigenvalues of $\boldsymbol{Q}$).

The state-variable vector is $\boldsymbol{x} = [p, r, \beta, \phi, \delta_r, \delta_a]^T$ and the control-variable vector is $\boldsymbol{u} = [\delta_{rc}, \delta_{ac}]^T$, where p, r, β, and ϕ represent the roll rate (rad/s), yaw rate (rad/s), sideslip angle (rad), and roll angle (rad); δ_r and δ_a are the deflection angles (rad) of the rudder and aileron; and δ_{rc} and δ_{ac} are the rudder and aileron command (rad), respectively. The open-loop eigenvalues are $-0.1034 \pm 2.093j$, -0.7828, and -0.006188 for the aircraft dynamics and -1 and -5 for the rudder and aileron actuator dynamics. The desired eigenvalues are given as $\Lambda^* = [-0.63 \pm 2.42j, -4, -0.05, -11, -5.05]^T$. The initial weighting matrix is chosen as $\boldsymbol{Q}_0 = \mathrm{diag}\{0.51, 0.51, 0.51, 0.51, 0.51, 0.76\}$. The weighting matrix $\boldsymbol{R} = \boldsymbol{I}_2$ is not changed while the CLEs are shifted. The CLEs for the weighting matrices $(\boldsymbol{Q}_0, \boldsymbol{R})$ are $\Lambda_0 = [-1.278 \pm 1.943j, -4.627, -1.012, -17.36, -8.789]^T$, which are linearly and simultaneously shifted to Λ^*. Two cases are considered: In the first case, eigenvalues of $\boldsymbol{Q}$ are not adjusted; and in the second case, they are adjusted. Figure 1 shows trajectories of the eigenvalues in the first case. One eigenvalue becomes negative, so that the weighting matrix obtained is not nonnegative definite. On the other hand, in the second case, as shown in Fig. 2, all of the eigenvalues take positive values throughout the integral contour. Three eigenvalues decreasing toward zero retain positive values by taking the procedure shown in the preceding section. In this example, the thresholds are given as 0, 0.01, and 0.02. Thus, the following weighting matrix is obtained:

$$\boldsymbol{Q} = \begin{bmatrix} 0.1089 & 0.1804 & 0.0239 & -0.0070 & -0.0225 & -0.0398 \\ 0.1804 & 0.3414 & 0.0541 & -0.0126 & -0.0350 & -0.0613 \\ 0.0239 & 0.0541 & 0.4970 & 0.0066 & 0.1075 & 0.1097 \\ -0.0070 & -0.0126 & 0.0066 & 0.0009 & 0.0052 & 0.0077 \\ -0.0225 & -0.0350 & 0.1075 & 0.0052 & 0.0511 & 0.0374 \\ -0.0398 & -0.0613 & 0.1097 & 0.0077 & 0.0374 & 0.1426 \end{bmatrix}$$

The eigenvalues of $\boldsymbol{Q}$ are 1.201×10^{-5}, 1.003×10^{-2}, 2.005×10^{-2}, 9.063×10^{-2}, 4.602×10^{-1}, and 5.610×10^{-1}, which means that $\boldsymbol{Q}$ is positive definite.

V. Conclusions

The proposed method provides a weighting matrix that achieves pole placement for the LQR while making the weighting matrix nonnegative definite by controlling its eigenvalues. The nonnegative definiteness of the weighting matrix ensures desirable properties of the regulator, such as large gain and phase margins. Therefore, the method is a useful pole-assignment method for multi-input state feedback control systems.

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Asymptotic Linear Quadratic Control for Lightly Damped Structures

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I. Introduction

LINEAR quadratic control has been used in many capacities for controlled structures problems because of its ease in design and insight it adds into problems such as robust control, structural design, and actuator/sensor selection. Campbell and Crawley[1] developed a set of rules for the design of low-order, robust compensators, called classically rationalized compensators, which utilizes a rigorous examination of the single-input single-output (SISO) linear quadratic Gaussian (LQG) compensator. Wie and Byun[2] compared certain classical design methods with the LQG compensator. And Crawley et al.[3] used both linear quadratic regulator (LQR) and LQG controllers for the preliminary design of a controlled structure.

In experimental applications, however, linear quadratic controllers suffer from important weaknesses such as high order and low robustness.[4] Many design methods attempt to address these concerns within the linear quadratic framework. This Note presents a general form for the LQR and LQG compensators designed for lightly damped structures, and examines the asymptotic properties of these controllers. The asymptotes add insight into how the linear quadratic controllers compensate a flexible system, which can be used to obtain classical design rules motivated by optimal control, improve the order and robustness weaknesses of these controllers, and provide a framework for adding insight to other design methods.

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II. Problem Statement

Consider a system

$$\dot{x} = Ax + B_u u + B_w w$$
$$y = C_y x + v, \qquad z = C_z x \tag{1}$$

in 2 × 2 block modal form, where the ith mode is given by

$$A_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta_i\omega_i \end{bmatrix}, \qquad x_i = \begin{bmatrix} q_i \\ \dot{q}_i \end{bmatrix}, \qquad B_{ui} = \begin{bmatrix} 0 \\ b_{ui} \end{bmatrix}$$

$$B_{wi} = \begin{bmatrix} 0 \\ b_{wi} \end{bmatrix}, \qquad C_{yi} = [c_{yqi} \quad c_{y\dot{q}i}], \qquad C_{zi} = [c_{zqi} \quad c_{z\dot{q}i}]$$

Note that w is a vector of exogenous inputs (disturbances), and v is a vector of sensor noises. The features of a structural system are given: stable, lightly damped, and modally dense.

In transfer function form, the system is given as

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} g_{zw} & g_{zu} \\ g_{yw} & g_{yu} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} \dfrac{n_{zw}}{d} & \dfrac{n_{zu}}{d} \\ \dfrac{n_{yw}}{d} & \dfrac{n_{yu}}{d} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{2}$$

III. LQR Controller

The LQR controller minimizes a quadratic cost with weightings on the states and inputs:

$$J_{\mathrm{LQR}} = \int_0^\infty \left(x^T R_{xx} x + u^T R_{uu} u\right) \mathrm{d}t \tag{3}$$

In this work, R_{xx} is chosen to penalize the performance and R_{uu} is chosen to weight the inputs equally, or

$$R_{xx} = C_z^T C_z \geq 0, \qquad R_{uu} = \rho \cdot I > 0 \tag{4}$$

where ρ is a positive scalar. Minimization of the LQR cost yields a matrix of optimal gains F for state feedback:

$$u = Fx = -R_{uu}^{-1} B_u^T P x \tag{5}$$

where P is the solution matrix of the algebraic Riccati equation (ARE)

$$PA + A^T P + R_{xx} - PB_u R_{uu}^{-1} B_u^T P = 0 \tag{6}$$

The Kalman filter minimizes the state estimation error where the disturbance and sensor noise are assumed to be zero mean, Gaussian processes that are uncorrelated in time. For this work, the covariances are chosen to be

$$E\{ww^T\} = I, \qquad E\{vv^T\} = \theta \cdot I > 0 \tag{7}$$

where θ is a positive scalar. Note that a dual ARE exists for the Kalman filter.

A. Expensive Control LQR Asymptote

To study the expensive control LQR problem, or as ρ tends to be large, the solution matrix P is expanded in powers of $\sqrt{\rho}$:

$$P = \sqrt{\rho} P_0 + P_1 + (1/\sqrt{\rho}) + \cdots \tag{8}$$

Substituting into Eq. (6), and collecting like powers,

$$\mathcal{O}(\sqrt{\rho}) \quad P_0 A + A^T P_0 = 0 \tag{9}$$

$$\mathcal{O}(1) \quad P_1 A + A^T P_1 + R_{xx} - P_0 B B^T P_0 = 0 \tag{10}$$